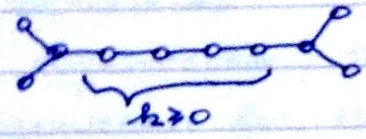
PROBLEM SET #1 (SPECTRAL GRAPH THEORY)

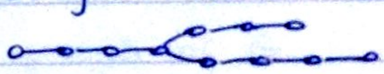
① Let L be the Laplacian matrix of G. Prove that
$$x^T L x = \sum_{uv \in E(G)} (x_u - x_v)^2.$$

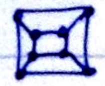
② Determine the adjacency eigenvalues of the n-cycle C_n.
Describe corresponding eigenvectors.

③ Prove that 

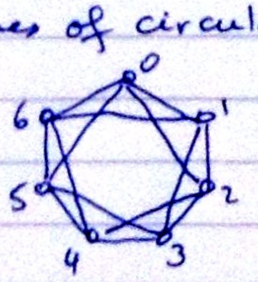
has eigenvalue 2.

Prove that this eigenvalue is λ_1.
Hint: Find the eigenvector.

④ Use interlacing for λ_1 to show that a graph with all eigenvalues in $(-2,2)$ is disjoint union of paths and graphs of the form:  (pending paths of various lengths)

⑤ Determine the eigenvalues (& their multiplicities) of the d-cube Q_d. If this is hard, start with
$Q_1 =$ , $Q_2 =$ , $Q_3 =$ .

⑥ Eigenvalues of circulants. Try special cases: $C_n(1,2)$.

$C_7(1,2)$ 

⑦ Determine Laplacian eigenvalues of P_n.

⑧ Show that Laplacian of G and its complement $\overline{G}$ have the same set of eigenvectors. Use this to express $\{\mu_j(\overline{G})\}$ in terms of $\{\mu_j(G)\}$.

(9) Find the eigenvalues of the Petersen graph (online help allowed). The find the eigenvalues of its line graph. (online help allowed.)

<u>Question:</u> Does the Hoffman formula give that Pet has no 3-edge-coloring? (why not?)

What about using some "weighted" matrix B?

(10) Suppose that G has an automorphism $\alpha : V(G) \to V(G)$.

~~[illegible]~~

If x is an eigenvector, prove that $x \circ \alpha : V \xrightarrow{\alpha} V \xrightarrow{x} \mathbb{R}$ is also an eigenvector for the same eigenvalue.

What can you say about x if the corresponding eigenvalue is simple?

(11) $L(G) = QQ^T$ (where Q is $V \times E$ ± 1-incidence matrix).

What can you say about the matrix Q^TQ?

Hint: Think of the line graph of G.

(Realize some relationship with problem #9.)